

On extension functions for image space with different separation axioms [☆]

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Abstract

In this paper we study a sufficient conditions for continuous and θ_α -continuous extensions of f to space X for an image space Y with different separation axioms.

Keywords: $S(n)$ -space, regular space, continuous function, θ_α -continuous function, $U(\alpha)$ -space, regular $U(\alpha)$ -space

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1. Introduction

It is work was performed as part of the general problem, which is as follows. Let f be a continuous mapping of a dense set S of the topological space X into the topological space Y . Required to find the necessary and sufficient conditions for continuous extension of f to the space X . (i.e. existence a continuous mapping $F : X \mapsto Y$ such that $F \upharpoonright S = f$). This problem can be considered more widely, if the continuous mapping is replaced by "almost" continuous. For example, we will consider the θ_α -continuous mapping.

First sufficient condition for continuous extension of f to the space X into the regular space Y was obtained by N. Bourbaki. In [5] was proved that this condition is not sufficient condition for no regular space Y .

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The necessary and sufficient conditions for continuous extension of f on the space X was obtained:

- in [6] for metrizable compact spaces Y ;
- in [4] for compact spaces Y ;
- in [3] for Lindelöf spaces Y ;
- in [8] for realcompact spaces Y ;
- in [3] for regular spaces Y .

So for a compact spaces Y we have the next result (see [4]).

Theorem 1.1. (*Taimanov*) *Let f be a continuous mapping of a dense set S of a topological space X into a compact space Y , then the following are equivalent:*

1. f to have a continuous extension to X ;
2. if A and B are disjoint closed subsets of Y then $\overline{f^{-1}(A)} \cap \overline{f^{-1}(B)} = \emptyset$.

Consider the following

• Condition (*): if a family $\{A_\beta\}$ of closed subsets of Y such that $\bigcap_\beta A_\beta = \emptyset$ implies $\bigcap_\beta \overline{f^{-1}(A_\beta)} = \emptyset$.

So general result for a regular space Y is the following theorem ([3]).

Theorem 1.2. (*Velichko*) *Let f be a continuous mapping of a dense set S of a topological space X into a regular space Y , then the following are equivalent:*

1. f to have a continuous extension to X ;
2. condition (*) holds.

Note that if Y is a Tychonoff space, then we can in condition (*) a closed subsets be replaced by zero-sets of Y .

Theorem 1.3. (*Velichko*) *Let f be a continuous mapping of a dense set S of the topological space X into a Lindelöf space Y , then the following are equivalent:*

1. f to have a continuous extension to X ;
2. for any sequence $\{A_i\}$ of zero-sets of Y such that $\bigcap_i A_i = \emptyset$ implies $\bigcap_i \overline{f^{-1}(A_i)} = \emptyset$.

Note that the condition $(*)$ is a necessary condition for continuous extension of f to X for any space Y .

Proposition 1.4. *Let f be have a continuous extension to X for a space Y . Then condition $(*)$ holds.*

Proof. Let F be a continuous extension to X for a space Y and $\{A_\beta\}$ be a family of closed subsets of Y such that $\bigcap_\beta A_\beta = \emptyset$. Fix $x \in X$. There is β such that $F(x) \notin A_\beta$, hence there exist a neighborhood V of $F(x)$ such that $V \cap A_\beta = \emptyset$. Since F is continuous map, $F^{-1}(V)$ is a neighborhood of x . It follows that $F^{-1}(V) \cap F^{-1}(A_\beta) = \emptyset$ and $x \notin \overline{f^{-1}(A_\beta)}$. \square

In this paper we study a sufficient conditions for continuous and θ_α -continuous extensions of f to X for an image space Y with different separation axioms.

P.S. Alexandroff and P.S. Urysohn [1] first defined R -closed spaces in 1924. U -closed (R -closed) spaces are Urysohn (regular) spaces, closed in any Urysohn (regular) space containing them. Recall that the solutions to the standard extension of continuous functions problem in the setting of R -closed or U -closed spaces are unknown.

Question 1 (Q.22 in [12]). Let X be an R -closed extension of a space S and $f : S \mapsto Y$ be a continuous function where Y is R -closed. Find a necessary and sufficient condition for f to have a continuous extension to X .

Question 2 (Q.23 in [12]). Let X be an U -closed extension of a space S and $f : S \mapsto Y$ be a continuous function where Y is U -closed. Find a necessary and sufficient condition for f to have a continuous extension to X .

In this paper we find a necessary and sufficient condition for f to have a continuous extension to X where Y is $U(\alpha)$ -space (regular $U(\alpha)$ -space).

2. Main definitions and notation

We say that U is a neighborhood of a set A if U is an open set in X such that $A \subseteq U$.

The closure of a set A will be denoted by \overline{A} , $[A]$ or $cl(A)$; the symbol \emptyset stands for the empty set. As usual, $f(A)$ and $f^{-1}(A)$ are the image and the complete preimage of the set A under the mapping f , respectively.

Let $\alpha > 0$ be an ordinal.

Definition 2.1. A neighborhood U of a set A is called an α -hull of the set A if there exists a set of neighborhoods $\{U_\beta\}_{\beta \leq \alpha}$ of the set A such that $clU_\beta \subseteq U_{\beta+1}$ for any $\beta + 1 \leq \alpha$ and $U = U_\alpha = \bigcup_{\beta \leq \alpha} U_\beta$.

For $\alpha = 1$, a 1-hull of the set A is an open set containing the set A .

Let X be a topological space, $M \subseteq X$, $x \in X$ and $\alpha > 0$ be an ordinal. We consider the θ^α -closure operator: $x \notin cl_{\theta^\alpha} M$ if there is an α -hull U of the point x such that $clU \cap M = \emptyset$ if $\alpha > 1$; $cl_{\theta^0} M = clM$ if $\alpha = 0$; and, for $\alpha = 1$, we get the θ -closure operator, i.e., $cl_{\theta^1} M = cl_\theta M$.

A set M is θ^α -closed if $M = cl_{\theta^\alpha} M$. Denote by $Int_{\theta^\alpha} M = X \setminus cl_{\theta^\alpha}(X \setminus M)$ the θ^α -interior of the set M . Evidently, $cl_{\theta^{\alpha_1}}(cl_{\theta^{\alpha_2}} M) = cl_{\theta^{\alpha_1 + \alpha_2}} M$ for $M \subseteq X$ and any α_1, α_2 ordinal numbers.

For $\alpha > 0$ and a filter \mathcal{F} on X , denote by $ad_{\theta^\alpha} \mathcal{F}$ the set of θ^α -adherent points, i.e., $ad_{\theta^\alpha} \mathcal{F} = \bigcap \{cl_{\theta^\alpha} F_\beta : F_\beta \in \mathcal{F}\}$. In particular, $ad_{\theta^0} \mathcal{F} = ad \mathcal{F}$ is the set of adherent points of the filter \mathcal{F} . For any α , a point $x \in X$ is $S(\alpha)$ -separated from a subset M if $x \notin cl_{\theta^\alpha} M$. For example, x is $S(0)$ -separated from M if $x \notin clM$. For $\alpha > 0$, the relation of $S(\alpha)$ -separability of points is symmetric. On the other hand, $S(0)$ -separability may be not symmetric in some not T_1 -spaces. Therefore, we say that points x and y are $S(0)$ -separated if $x \notin cl_X \{y\}$ and $y \notin cl_X \{x\}$.

Let $n \in \mathbb{N}$ and X be a topological space.

1. X is called an $S(n)$ -space if any two distinct points of X are $S(n)$ -separated.

Next, we define a series of separation axioms for $\alpha > 0$.

2. X will be called an $U(\alpha)$ -space if any two distinct points x and y of X there are U_x, U_y an α -hull of x and y such that $\overline{U_x} \cap \overline{U_y} = \emptyset$.

Note that a regular space is a $S(n)$ -($U(n)$ -)space for any $n \in \mathbb{N}$, and a functionally Hausdorff space is a $U(\omega)$ -space.

3. X will be called a regular- $U(\alpha)$ -space if for a point x and a closed set F such that $x \notin F$ there are U_x, U_F an α -hull of x and F such that $\overline{U_x} \cap \overline{U_F} = \emptyset$.

Note that a Tychonoff space is a regular- $U(\omega)$ -space.

It is obvious that $S(0)$ -spaces are T_1 -spaces, $S(1)$ -spaces are Hausdorff spaces, and $S(2)$ -spaces are Urysohn spaces.

A set of all of neighborhoods of x will be denoted by $\mathcal{N}(x)$.

A set of all α -hull of the set A will be denoted by $\mathcal{N}_{\theta^\alpha}(A)$.

3. Continuous extension

Definition 3.1. Let X, Y be a topological spaces, S be a dense subset of X , f be a continuous map from S into Y and V be a subset of Y . A point $x \in X$ will be called X_{θ^α} -interior point of $f^{-1}(V)$, if

$$\bigcap \{[f(P \cap S)]_{\theta^\alpha} : P \in \mathcal{N}(x)\} \subseteq V \text{ holds.}$$

Set of all X_{θ^α} -interior points of $f^{-1}(V)$ will be denoted by $X_{\theta^\alpha}(f^{-1}(V))$.

Proposition 3.2. Let X be a topological space, Y be a $U(\alpha)$ -space, S be a dense subset of X , f be a continuous map from S into Y , V be a open subset of Y , $x \in X$ and the condition $(*)$ holds. Then the set $\bigcap \{[f(P \cap S)]_{\theta^\alpha} : P \in \mathcal{N}(x)\} = \{p\}$ for some $p \in Y$.

Proof. Note that $x \in \bigcap \{\overline{P \cap S} : P \in \mathcal{N}(x)\}$. By condition $(*)$, $\bigcap \{\overline{f(P \cap S)} : P \in \mathcal{N}(x)\} \neq \emptyset$ and, hence, $T = \bigcap \{[f(P \cap S)]_{\theta^\alpha} : P \in \mathcal{N}(x)\} \neq \emptyset$. Let $y \in T$ and $z \neq y$. By $U(\alpha)$ -separateness of Y , there are α -hulls $O(y)$ and $O(z)$ such that $\overline{O(y)} \cap \overline{O(z)} = \emptyset$. Then $\overline{f^{-1}(\overline{O(y)})} \cap \overline{f^{-1}(\overline{O(z)})} = \emptyset$. Note, that $x \in \overline{f^{-1}(\overline{O(y)})}$. Then there is $W \in \mathcal{N}(x)$ such that $W \cap \overline{f^{-1}(\overline{O(z)})} = \emptyset$. It follows that $f(W \cap S) \cap \overline{O(z)} = \emptyset$ and $z \notin [f(W \cap S)]_{\theta^\alpha}$. □

• Condition $(*_\alpha)$: for each an open set V of Y the set $X_{\theta^\alpha}(f^{-1}(V))$ is open set of X .

Theorem 3.3. Let X be a topological space, Y be a $U(\alpha)$ -space, S be a dense subset of X , f be a continuous map from S into Y , then the following are equivalent:

1. f to have a continuous extension to X ;
2. conditions $(*)$ and $(*_\alpha)$ holds.

Proof. (1) \Rightarrow (2). From Proposition 1.4, we obtain condition $(*)$. Let f to have a continuous extension F to X and let V be an open set of Y . We prove that $X_{\theta^\alpha}(f^{-1}(V)) = F^{-1}(V)$.

Let $x \in X_{\theta^\alpha}(f^{-1}(V))$. Since F is a continuous map and the filter base $\{P \cap S : P \in \mathcal{N}(x)\}$ converges to a point x , it follows that the filter base $\mathcal{F} = \{f(P \cap S) : P \in \mathcal{N}(x)\}$ converges to a point $F(x)$. By $U(\alpha)$ -separateness of Y , $ad_{\theta^\alpha} \mathcal{F} = F(x)$, hence, $F(x) \in V$ and $x \in F^{-1}(V)$.

Let $x \in F^{-1}(V)$. Then $F(x) \in V$. By unique of adherent point of the filter base $\mathcal{F} = \{f(P \cap S) : P \in \mathcal{N}(x)\}$, we have $ad_{\theta\alpha}\mathcal{F} = F(x) \in V$ and $x \in X_{\theta\alpha}(f^{-1}(V))$.

(2) \Rightarrow (1). For each point $x \in X$ consider $F(x) = \bigcap \{[f(P \cap S)]_{\theta\alpha} : P \in \mathcal{N}(x)\}$. By proposition 3.2, $F(x)$ is an unique point of Y . We have the map $F : X \mapsto Y$. Note that if $x \in S$ then $F(x) = f(x)$. Clearly that $x \in P \cap S$ for any $P \in \mathcal{N}(x)$, so $f(x) \in \bigcap \{f(P \cap S) : P \in \mathcal{N}(x)\} \subseteq \bigcap \{[f(P \cap S)]_{\theta\alpha} : P \in \mathcal{N}(x)\} = F(x)$. We have that F is an extension f on X . We claim that F is a continuous extension on X . Let V be an open set of Y . By condition $(*_\alpha)$, $X_{\theta\alpha}(f^{-1}(V))$ is open set of X . It remains to prove that $X_{\theta\alpha}(f^{-1}(V)) = F^{-1}(V)$.

Let $x \in X_{\theta\alpha}(f^{-1}(V))$. Then, by condition $(*_\alpha)$, $F(x) = \bigcap \{[f(P \cap S)]_{\theta\alpha} : P \in \mathcal{N}(x)\} \subseteq V$, and $x \in F^{-1}(V)$.

Let $x \in F^{-1}(V)$. Then $F(x) \in V$ and $\bigcap \{[f(P \cap S)]_{\theta\alpha} : P \in \mathcal{N}(x)\} = F(x) \in V$, thus the point x is $X_{\theta\alpha}$ -interior point of $f^{-1}(V)$, hence, $x \in X_{\theta\alpha}(f^{-1}(V))$. It follow that $X_{\theta\alpha}(f^{-1}(V)) = F^{-1}(V)$. □

Corollary 3.4. Let X be a topological space, Y be a Urysohn space, S be a dense subset of X , f be a continuous map from S into Y , then the following are equivalent:

1. f to have a continuous extension to X ;
2. for each an open set V of Y the set $X(f^{-1}(V))$ is open set of X and condition $(*)$ holds.

Corollary 3.5. Let X be a topological space, Y be a functionally Hausdorff space, S be a dense subset of X , f be a continuous map from S into Y , then the following are equivalent:

1. f to have a continuous extension to X ;
2. conditions $(*)$ and $(*_\omega)$ holds.

Proposition 3.6. Let Y be a regular- $U(\alpha)$ -space. Then $(*)$ implies $(*_\alpha)$.

Proof. Let S be a dense subset of X , $f : S \mapsto Y$ be a continuous function and condition $(*)$ holds. We prove that a set $X_{\theta\alpha}(f^{-1}(V))$ is open set of X for an open set V of Y . Let $x \in X_{\theta\alpha}(f^{-1}(V))$. By proposition 3.2, $F(x) = \bigcap \{[f(P \cap S)]_{\theta\alpha} : P \in \mathcal{N}(x)\}$ is an unique point of Y . As Y is a regular- $U(\alpha)$ -space there is α -hull W of point $F(x)$ and α -hull H of set $X \setminus V$

such that $\overline{W} \cap \overline{H} = \emptyset$. Let $\gamma = \{TP = [f(P \cap S) \cap (X \setminus \overline{W})]_{\theta^\alpha} : P \in \mathcal{N}(x)\}$. Then $\bigcap \gamma = \emptyset$ and $\bigcap \{f^{-1}(TP) : P \in \mathcal{N}(x)\} = \emptyset$. There is a neighborhood U of x such that $U \cap f^{-1}(TP) = \emptyset$ for some $P \in \mathcal{N}(x)$. Let $Q = U \cap P$. Then $[f(P \cap S)]_{\theta^\alpha} \subseteq \overline{W}$. It follows that $Q \subseteq X_{\theta^\alpha}(f^{-1}(V))$ and, hence, the set $X_{\theta^\alpha}(f^{-1}(V))$ is an open set of X . □

Theorem 3.7. *Let X be a topological space, Y be a regular- $U(\alpha)$ -space, S be a dense subset of X , f be a continuous map from S into Y , then the following are equivalent:*

1. *f to have a continuous extension to X ;*
2. *conditions $(*)$ holds.*

4. θ_α -continuous extension

Recall that a function $f : X \mapsto Y$ be called θ -continuous if for a point $x \in X$ and a neighborhood U of $f(x)$ there is a neighborhood W of x such that $f(W) \subseteq \overline{U}$.

Definition 4.1. A function $f : X \mapsto Y$ will be called θ_α -continuous if for a point $x \in X$ and a α -hull U of $f(x)$ there is a neighborhood W of x such that $f(W) \subseteq \overline{U}$.

For $\alpha = 1$, we have that θ_1 -continuous function is θ -continuous function.

Clearly, that a continuous function is a θ_α -continuous function for any $\alpha > 0$. Moreover, it is easy to check that θ_β -continuous function is θ_α -continuous function for $\beta < \alpha$.

- Condition $(+)_\alpha$: a family $\{A_\beta\}$ of subsets of Y such that $\bigcap_\beta [A_\beta]_{\theta^\alpha} = \emptyset$ implies $\bigcap_\beta \overline{f^{-1}(A_\beta)} = \emptyset$.

- Condition $(++)_\alpha$: for each α -hull $W = \bigcup_{\beta \leq \alpha} U_\beta$ of a point $y \in Y$ there is an open set V of X such that $X_{\theta^\alpha}(f^{-1}(U_1)) \subseteq V \subseteq X_{\theta^\alpha}(f^{-1}(\overline{W}))$.

Theorem 4.2. *Let X be a topological space, Y be a $U(\alpha)$ -space, S be a dense subset of X , f be a θ_α -continuous map from S into Y , then the following are equivalent:*

1. f to have a θ_α -continuous extension to X ;
2. conditions $(+)_\alpha$ and $(++)_\alpha$ holds.

Proof. (1) \Rightarrow (2). Let F be a θ_α -continuous extension of f , a family $\sigma = \{A_\beta\}$ such that $\bigcap_\beta [A_\beta]_{\theta^\alpha} = \emptyset$, $x \in X$ and $y = F(x)$. There is $B \in \sigma$ such that $y \notin [B]_{\theta^\alpha}$ and, hence, there is a α -hull W of y such that $\overline{W} \cap B = \emptyset$. There exists a neighborhood V of x such that $\overline{F(V)} \subseteq \overline{W}$. Since $\overline{W} \cap B = \emptyset$, we get $V \cap f^{-1}(B) = \emptyset$, and, hence, $x \notin \overline{f^{-1}(B)}$. It follows that $\bigcap \{\overline{f^{-1}(B)} : B \in \sigma\} = \emptyset$. So we have condition $(+)_\alpha$ holds.

Let $W = \bigcup_{\beta \leq \alpha} U_\beta$ (where $cl U_\beta \subseteq U_{\beta+1}$ for any $\beta + 1 \leq \alpha$ and $W = U_\alpha = \bigcup_{\beta \leq \alpha} U_\beta$) be an α -hull of some point of Y and $x \in X$ such that

$x \in X_{\theta^\alpha}(f^{-1}(U_1))$. Since $x \in \bigcap \{\overline{P \cap S} : P \in \mathcal{N}(x)\}$ we get $F(x) \in \bigcap \{[f(P \cap S)]_{\theta^\alpha} : P \in \mathcal{N}(x)\}$. If $a \neq F(x)$ then there are a α -hull $O(a)$ and $O(F(x))$ such that $\overline{O(a)} \cap \overline{O(F(x))} = \emptyset$. Then there is a neighborhood P of x such that $F(P) \subseteq O(F(x))$. It follow that $a \notin [f(P \cap S)]_{\theta^\alpha}$ and $\bigcap \{[f(P \cap S)]_{\theta^\alpha} : P \in \mathcal{N}(x)\} = F(x)$. Since $F(x) \in U_1$ (and W is α -hull of $F(x)$) there is a neighborhood V_x of x such that $F(V_x) \subseteq \overline{W}$. So if $z \in V_x$ then $F(z) \in \overline{W}$ and, hence, $V_x \subseteq X_{\theta^\alpha}(f^{-1}(\overline{W}))$. Let $V = \bigcup \{V_x : x \in X_{\theta^\alpha}(f^{-1}(W))\}$.

So we have condition $(++)_\alpha$ holds.

(2) \Rightarrow (1). Let $F(x) := \bigcap \{[f(P \cap S)]_{\theta^\alpha} : P \in \mathcal{N}(x)\}$. By condition $(+)_\alpha$, $F(x) \neq \emptyset$. We claim that $F(x)$ is an unique point. Let $y \in F(x)$ and $z \neq y$. Then there are α -hull $O(y)$ and $O(z)$ of points y and z such that $\overline{O(y)} \cap \overline{O(z)} = \emptyset$. Let P be a neighborhood of x . Then $z \notin [f(P \cap S) \cap \overline{O(y)}]_{\theta^\alpha}$. We claim that $z \notin F(x)$. On the contrary, let $z \in F(x)$. Then a family $\gamma = \{f(P \cap S) \cap \overline{P(z)} : P \in \mathcal{N}(x), P(z) \in \mathcal{N}_{\theta^\alpha}(z)\}$ consists of a non-empty sets and $y \notin [f(P \cap S) \cap \overline{O(z)}]_{\theta^\alpha}$. Consider a family $\sigma = \gamma \cup \{f(P \cap S) \cap \overline{P(y)} : P \in \mathcal{N}(x), P(y) \in \mathcal{N}_{\theta^\alpha}(y)\}$.

We claim that $D = \bigcap \{[B]_{\theta^\alpha} : B \in \sigma\} = \emptyset$. So $y \notin D$ and $z \notin D$. Let $q \in Y \setminus \{y, z\}$. There are α -hull $P(y)$ and $P(q)$ of points y and q such that $\overline{P(y)} \cap \overline{P(q)} = \emptyset$. Then $q \notin [f(P \cap S) \cap \overline{P(y)}]_{\theta^\alpha}$ and $q \notin D$. So $D = \emptyset$.

By condition $(+)_\alpha$, $\bigcap \{f^{-1}(B) : B \in \sigma\} = \emptyset$. Hence, there is $C \in \sigma$ such that $x \notin \overline{f^{-1}(C)}$. Note that $x \in \overline{f^{-1}(B)}$ for any $B \in (\sigma \setminus \gamma)$ (by definition of $F(x)$). It follows that $C \in \gamma$, i.e. $C = f(P \cap S) \cap \overline{P(z)}$ for some $P \in \mathcal{N}(x)$ and $P(z) \in \mathcal{N}_{\theta^\alpha}(z)$. There is a neighborhood Q of x such that $Q \cap f^{-1}(C) = \emptyset$ and $Q \subseteq P$. So $f(Q \cap S) \cap \overline{P(z)} = \emptyset$ and $z \notin F(x)$.

So we get extension F of the map f . We claim that F is a θ_α -continuous extension to X .

Let $x \in X$ and $W = \bigcup_{\beta \leq \alpha} U_\beta$ be a α -hull of $F(x)$. Then $F(x) = \bigcap \{[f(P \cap S)]_{\theta^\alpha} : P \in \mathcal{N}(x)\}$ and $F(x) \in U_1$. Hence $x \in X_{\theta^\alpha}(f^{-1}(U_1))$. By condition $(++)_\alpha$, there is an open set V of X such that $X_{\theta^\alpha}(f^{-1}(U_1)) \subseteq V \subseteq X_{\theta^\alpha}(f^{-1}(\overline{W}))$. It follows that $F(V) \subseteq \overline{W}$. \square

Note that for $\alpha = 1$ we get

- condition $(+)$: a family $\{A_\beta\}$ of subsets of Y such that $\bigcap_\beta [A_\beta]_\theta = \emptyset$ implies $\bigcap_\beta \overline{f^{-1}(A_\beta)} = \emptyset$.
- condition $(++)$: for each open set W of X there is an open set V of X such that $X_\theta(f^{-1}(W)) \subseteq V \subseteq X_\theta(f^{-1}(\overline{W}))$.

Corollary 4.3. Let X be a topological space, Y be a Urysohn space, S be a dense subset of X , f be a θ -continuous map from S into Y , then the following are equivalent:

1. f to have a θ -continuous extension to X ;
2. conditions $(+)$ and $(++)$ holds.

Note that for a regular- $U(\alpha)$ -space Y a θ_α -continuous function is a continuous function and $cl_{\theta^\alpha} M = cl M$. It follows that a condition $(+)_\alpha$ is equivalent to the condition $(*)$ and we get a Theorem 3.7.

5. Example

There is a simple example of a regular space, but it is not completely regular space (see [10]).

Example 5.1. (*Mysior*) Let M_0 be the subset of the plane defined by the condition $y \geq 0$, i.e., the closed upper half-plane, let z_0 be the point $(0, -1)$ and let $M = M_0 \cup \{z_0\}$. Denote by L the line $y = 0$ and by L_i where $i = 1, 2, \dots$, the segment consisting of all points $(x, 0) \in L$ with $i - 1 \leq x \leq i$. For each point $z = (x, 0) \in L$ denote by $A_1(z)$ the set of all points $(x, y) \in M_0$, where $0 \leq y \leq 2$, by $A_2(z)$ the set of all points $(x + y, y) \in M_0$, where $0 \leq y \leq 2$, and let $B(z)$ be the family of all sets of the form $(A_1(z) \cup A_2(z)) \setminus B$, where B is a finite set such that $z \notin B$. Furthermore, for each point $z \in M_0 \setminus L$ let $B(z) = \{\{z\}\}$ and, finally, let $B(z_0) = \{U_i(z_0)\}_{i=1}^\infty$, where $U_i(z_0)$ consists of z_0 and all points $(x, y) \in M_0$ with $x \geq i$.

It is well-known that the space M is a regular space, but it is not a Tychonoff space. Moreover, the space M is not regular $U(\omega)$ -space.

Let T be the space M , but a base of the point z_0 we define as $B(z_0) = M \setminus D$, where D a clopen subset of M such that $z_0 \notin D$. Note that the identity map $id : M \mapsto T$ is the Tychonoff functor.

Consider a continuous identity map $f : M_0 \mapsto M_0$ as the map from a dense subset M_0 of the space T into the space M .

1. f have not a θ_α -continuous extension to X for any $\alpha < \omega$.

Really, the set L_i is a θ_α -closed subset of M for any $i \in \mathbb{N}$ and $\alpha < \omega$. Clearly that $\bigcap L_i = \emptyset$, but $\bigcap_i \overline{f^{-1}(L_i)} = \{z_0\}$. This contradicts the condition $(+)_\alpha$ for $\alpha < \omega$.

2. f have a θ_ω -continuous extension to X .

Let $F = id : T \mapsto M$. Note that for a ω -hull W of z_0 of the space M , $L \subseteq W$. Hence, $F^{-1}(W)$ is an open set of T .

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